

Multiconductor Transmission Lines

From Maxwell's Equations to Per-Unit-Length RLGC Matrices and Lumped Discretization Models

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Technical Monograph and Reference Guide

Abstract

This document presents a unified mathematical and physical formulation of multiconductor transmission lines (MTLs), bridging fundamental electromagnetic field theory with practical high-speed circuit simulation. First, we derive the coupled multiconductor Telegrapher equations directly from Maxwell's curl equations under the Transverse Electromagnetic (TEM) mode assumption, demonstrating the path-independence of line voltages and the exact duality between transverse electrostatic and magnetostatic boundary-value problems. Second, we establish the definitions, physical loop and transverse interpretations, and standard matrix forms of the four $n \times n$ per-unit-length (PUL) parameter matrices: inductance $\mathbf{L}$, resistance $\mathbf{R}$, conductance $\mathbf{G}$, and capacitance $\mathbf{C}$, along with the fundamental TEM identities $\mathbf{LC} = \mu\epsilon \mathbf{I}_n$ and $\mathbf{G} = (\sigma/\epsilon)\mathbf{C}$. Third, we formulate the principles of spatial discretization of the Telegrapher partial differential equations, analyzing the canonical lumped Π , T, and Half-T (inverted-L) cell topologies used in SPICE-compatible transient circuit simulation. Finally, we provide signal integrity engineering rules of thumb for estimating the maximum significant frequency content of a digital pulse ($f_{\max} \approx 0.35/t_r$) and determining the minimum number of cascade segments N required to avoid artificial numerical dispersion while faithfully capturing distributed wave propagation.

Keywords: Electromagnetic Theory, Multiconductor Transmission Lines, Maxwell's Equations, Transverse Electromagnetic (TEM) Mode, Telegrapher Equations, Lumped Π /T/Half-T Models, Per-Unit-Length Parameters, Digital Pulse Bandwidth, Numerical Dispersion, Signal Integrity.

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1 Multiconductor Transmission Line (MTL) System

Consider a uniform multiconductor transmission line consisting of n signal conductors and an additional conductor, designated conductor 0 (the reference or return conductor, often earth ground or a chassis plane) [1]. The line cross section and surrounding dielectric medium are assumed uniform along the longitudinal propagation axis z .

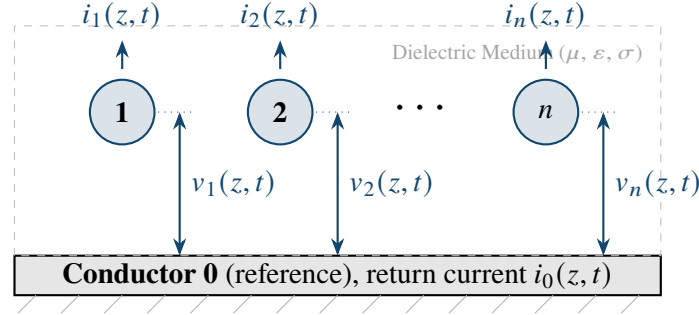


Figure 1. Geometry of a uniform multiconductor transmission line with n signal conductors and a reference conductor (conductor 0).

The line voltages of the n signal conductors with respect to the reference conductor are represented by the $n \times 1$ voltage vector $\mathbf{V}(z, t)$, and the signal conductor currents are represented by the $n \times 1$ current vector $\mathbf{I}(z, t)$:

$$\mathbf{V}(z, t) = \begin{bmatrix} v_1(z, t) \\ v_2(z, t) \\ \vdots \\ v_n(z, t) \end{bmatrix}, \quad \mathbf{I}(z, t) = \begin{bmatrix} i_1(z, t) \\ i_2(z, t) \\ \vdots \\ i_n(z, t) \end{bmatrix}. \quad (1.1)$$

By Kirchhoff's Current Law, the total return current flowing through the reference conductor 0 is:

$$i_0(z, t) = - \sum_{i=1}^n i_i(z, t). \quad (1.2)$$

2 Derivation of the Telegrapher Equations from Maxwell's Equations

We now derive the coupled transmission line equations directly from Maxwell's equations under the assumption of Transverse Electromagnetic (TEM) wave propagation.

2.1 Maxwell's Field Equations and the TEM Assumption

In a linear, isotropic, homogeneous medium with permeability μ , permittivity ε , and conductivity σ , the electromagnetic fields satisfy Maxwell's curl equations:

$$\nabla \times \vec{\mathcal{E}}(x, y, z, t) = -\frac{\partial \vec{\mathcal{B}}(x, y, z, t)}{\partial t} = -\mu \frac{\partial \vec{\mathcal{H}}(x, y, z, t)}{\partial t}, \quad (2.1)$$

$$\nabla \times \vec{\mathcal{H}}(x, y, z, t) = \vec{\mathcal{J}}(x, y, z, t) + \frac{\partial \vec{\mathcal{D}}(x, y, z, t)}{\partial t} = \sigma \vec{\mathcal{E}}(x, y, z, t) + \varepsilon \frac{\partial \vec{\mathcal{E}}(x, y, z, t)}{\partial t}, \quad (2.2)$$

supplemented by Gauss's laws $\nabla \cdot \vec{\mathcal{D}} = \rho_v$ and $\nabla \cdot \vec{\mathcal{B}} = 0$.

The TEM Mode Condition. A TEM mode along the longitudinal axis z is characterized by the complete absence of longitudinal electric and magnetic field components in the dielectric medium:

$$\mathcal{E}_z(x, y, z, t) = 0, \quad \mathcal{H}_z(x, y, z, t) = 0. \quad (2.3)$$

Consequently, the electric and magnetic fields reside entirely within the transverse cross-sectional (x - y) plane:

$$\begin{aligned} \vec{\mathcal{E}}(x, y, z, t) &= \vec{\mathcal{E}}_t(x, y, z, t) = \mathcal{E}_x \hat{x} + \mathcal{E}_y \hat{y}, \\ \vec{\mathcal{H}}(x, y, z, t) &= \vec{\mathcal{H}}_t(x, y, z, t) = \mathcal{H}_x \hat{x} + \mathcal{H}_y \hat{y}. \end{aligned} \quad (2.4)$$

Transverse Irrotationality and Path-Independent Voltage. Decomposing the del operator into transverse and longitudinal parts, $\nabla = \nabla_t + \hat{z} \frac{\partial}{\partial z}$ with $\nabla_t = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y}$, the longitudinal component of Faraday's law (2.1) gives:

$$\left[\nabla \times \vec{\mathcal{E}} \right]_z = \left(\nabla_t \times \vec{\mathcal{E}}_t \right) \cdot \hat{z} = -\mu \frac{\partial \mathcal{H}_z}{\partial t} = 0. \quad (2.5)$$

Because $\nabla_t \times \vec{\mathcal{E}}_t = \mathbf{0}$ at every cross section, the transverse electric field is conservative in every transverse plane. Consequently, $\vec{\mathcal{E}}_t$ can be expressed uniquely as the gradient of a scalar electric potential $\Phi(x, y, z, t)$:

$$\vec{\mathcal{E}}_t(x, y, z, t) = -\nabla_t \Phi(x, y, z, t). \quad (2.6)$$

Equation (2.6) is the fundamental prerequisite of transmission line theory: it guarantees that the voltage $v_i(z, t)$ of signal conductor i with respect to reference conductor 0 is strictly path-independent along any transverse path C_{0i} connecting them at longitudinal position z :

$$v_i(z, t) = \int_{\text{cond } 0}^{\text{cond } i} \vec{\mathcal{E}}_t \cdot d\vec{l} = \Phi(\text{cond } i, z, t) - \Phi(\text{cond } 0, z, t). \quad (2.7)$$

Similarly, by Ampère's law, the current $i_i(z, t)$ on conductor i is uniquely determined by integrating $\vec{\mathcal{H}}_t$ around any closed transverse contour Γ_i enclosing conductor i :

$$i_i(z, t) = \oint_{\Gamma_i} \vec{\mathcal{H}}_t \cdot d\vec{l}. \quad (2.8)$$

2.2 Derivation of the First Telegrapher Equation (Faraday's Law)

Consider an elementary transmission line section of length Δz spanning from z to $z + \Delta z$. For each signal conductor i ($i = 1, \dots, n$), define an oriented rectangular closed loop C_i spanning the longitudinal interval $[z, z + \Delta z]$ between reference conductor 0 and signal conductor i , as depicted in Figure 2.

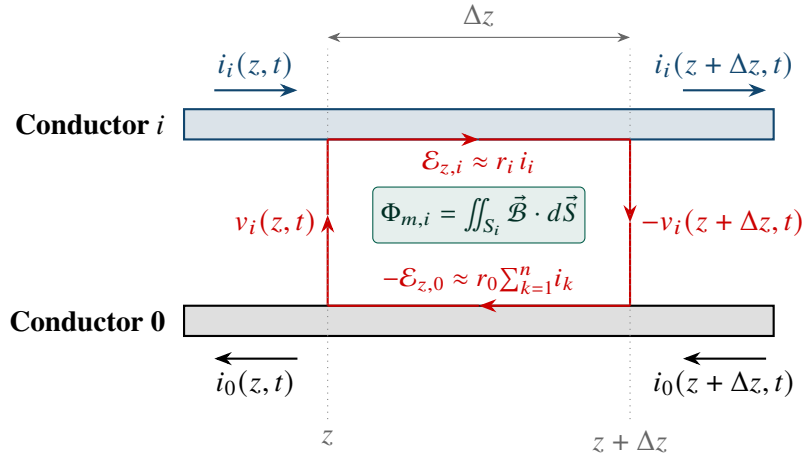


Figure 2. Faraday's law integration loop C_i along an incremental length Δz formed between signal conductor i and reference conductor 0.

Applying Faraday's law in integral form around contour C_i :

$$\oint_{C_i} \vec{\mathcal{E}} \cdot d\vec{l} = -\frac{\partial}{\partial t} \iint_{S_i} \vec{\mathcal{B}} \cdot d\vec{S} = -\frac{\partial \Phi_{m,i}(z, t)}{\partial t}, \quad (2.9)$$

where S_i is the planar surface bounded by C_i , and $\Phi_{m,i}$ is the total magnetic flux linking loop i . Evaluating the line integral along each of the four segments:

(a) *Transverse leg at z* : From conductor 0 to conductor i :

$$\int_{\text{cond } 0}^{\text{cond } i} \vec{\mathcal{E}}_t(x, y, z, t) \cdot d\vec{l} = -v_i(z, t). \quad (2.10)$$

(b) *Longitudinal leg along conductor i* : In real conductors with finite conductivity σ_c (the quasi-TEM approximation), an axial electric field $\mathcal{E}_{z,i}$ exists at the conductor surface to sustain current, characterized by the per-unit-length resistance r_i :

$$\int_z^{z+\Delta z} \mathcal{E}_{z,i} dz \approx r_i i_i(z, t) \Delta z. \quad (2.11)$$

(c) *Transverse leg at $z + \Delta z$* : From conductor i back to conductor 0:

$$\int_{\text{cond } i}^{\text{cond } 0} \vec{\mathcal{E}}_t(x, y, z + \Delta z, t) \cdot d\vec{l} = +v_i(z + \Delta z, t). \quad (2.12)$$

(d) *Longitudinal leg along reference conductor 0*: From $z + \Delta z$ back to z . Using $i_0(z, t) = -\sum_{k=1}^n i_k(z, t)$ from (1.2), the axial field is $\mathcal{E}_{z,0} \approx r_0 i_0(z, t) = -r_0 \sum_{k=1}^n i_k(z, t)$, yielding:

$$-\int_z^{z+\Delta z} \mathcal{E}_{z,0} dz \approx r_0 \left(\sum_{k=1}^n i_k(z, t) \right) \Delta z. \quad (2.13)$$

Summing all four segments in (2.9):

$$[v_i(z + \Delta z, t) - v_i(z, t)] + r_i i_i(z, t) \Delta z + r_0 \left(\sum_{k=1}^n i_k(z, t) \right) \Delta z = -\frac{\partial \Phi_{m,i}(z, t)}{\partial t}. \quad (2.14)$$

In a linear magnetic medium, the total per-unit-length magnetic flux linking loop i is a linear superposition of all line currents:

$$\lim_{\Delta z \rightarrow 0} \frac{\Phi_{m,i}(z, t)}{\Delta z} = \sum_{j=1}^n l_{ij} i_j(z, t), \quad (2.15)$$

where l_{ii} is the loop self-inductance per unit length and l_{ij} ($j \neq i$) is the mutual inductance per unit length between loops i and j . Dividing (2.14) by Δz and taking the limit $\Delta z \rightarrow 0$:

$$\frac{\partial v_i(z, t)}{\partial z} = - \left(r_i i_i(z, t) + r_0 \sum_{k=1}^n i_k(z, t) \right) - \sum_{j=1}^n l_{ij} \frac{\partial i_j(z, t)}{\partial t}. \quad (2.16)$$

Stacking all n equations for $i = 1, \dots, n$ into vector-matrix form directly produces the first Telegrapher equation:

$$\frac{\partial \mathbf{V}(z, t)}{\partial z} = -\mathbf{R}\mathbf{I}(z, t) - \mathbf{L} \frac{\partial \mathbf{I}(z, t)}{\partial t}. \quad (2.17)$$

2.3 Derivation of the Second Telegrapher Equation (Ampère-Maxwell Law)

To derive the differential equation for current variation along the line, consider an incremental length Δz of signal conductor i enclosed by a cylindrical control volume V_i of perimeter P_i . Applying charge conservation, which follows directly by taking the divergence of Ampère-Maxwell's law (2.2) ($\nabla \cdot (\nabla \times \vec{\mathcal{H}}) = 0$):

$$\nabla \cdot \vec{\mathcal{J}}_{\text{tot}} = \nabla \cdot \left(\vec{\mathcal{J}} + \frac{\partial \vec{\mathcal{D}}}{\partial t} \right) = 0. \quad (2.18)$$

Integrating over the volume V_i and invoking the divergence theorem:

$$\oint_{\partial V_i} \left(\vec{\mathcal{J}} + \frac{\partial \vec{\mathcal{D}}}{\partial t} \right) \cdot d\vec{S} = 0. \quad (2.19)$$

The closed surface ∂V_i consists of:

- *End faces at z and $z + \Delta z$:* The net longitudinal current exiting the wire segment is:

$$i_i(z + \Delta z, t) - i_i(z, t) = \frac{\partial i_i(z, t)}{\partial z} \Delta z. \quad (2.20)$$

- *Lateral perimeter surface $P_i \times [z, z + \Delta z]$:* The current escaping transversely into the surrounding dielectric medium is:

$$\oint_{P_i} \left(\vec{\mathcal{J}}_t + \frac{\partial \vec{\mathcal{D}}_t}{\partial t} \right) \cdot \hat{n} dl \Delta z, \quad (2.21)$$

where $\hat{n}$ is the outward transverse unit normal vector to conductor i .

Hence, by (2.19):

$$-\frac{\partial i_i(z, t)}{\partial z} \Delta z = \oint_{P_i} \vec{\mathcal{J}}_t \cdot \hat{n} dl \Delta z + \frac{\partial}{\partial t} \oint_{P_i} \vec{\mathcal{D}}_t \cdot \hat{n} dl \Delta z. \quad (2.22)$$

Substituting Ohm's law $\vec{\mathcal{J}}_t = \sigma \vec{\mathcal{E}}_t$ and the constitutive relation $\vec{\mathcal{D}}_t = \epsilon \vec{\mathcal{E}}_t$:

- (a) *Transverse conduction current (conductance $\mathbf{G}$):* The leakage current per unit length escaping from conductor i through the conducting dielectric to conductor 0 and all other signal conductors k depends on the transverse voltage differences:

$$\oint_{P_i} \sigma \vec{E}_t \cdot \hat{n} dl = g_{i0} v_i(z, t) + \sum_{\substack{k=1 \\ k \neq i}}^n g_{ik} [v_i(z, t) - v_k(z, t)] = \sum_{j=1}^n G_{ij} v_j(z, t), \quad (2.23)$$

where g_{i0} is the conductance to ground, g_{ik} is the mutual conductance between conductors i and k , and G_{ij} are the elements of the per-unit-length conductance matrix $\mathbf{G}$.

- (b) *Transverse displacement current (capacitance $\mathbf{C}$):* By Gauss's law in the transverse plane, the total free charge per unit length on conductor i is $q_i(z, t) = \oint_{P_i} \epsilon \vec{E}_t \cdot \hat{n} dl$. In the electrostatic transverse plane, this charge is linearly related to the conductor voltages:

$$q_i(z, t) = c_{i0} v_i(z, t) + \sum_{\substack{k=1 \\ k \neq i}}^n c_{ik} [v_i(z, t) - v_k(z, t)] = \sum_{j=1}^n C_{ij} v_j(z, t), \quad (2.24)$$

where c_{i0} is the partial capacitance to ground, c_{ik} is the mutual capacitance between conductors i and k , and C_{ij} are the elements of the per-unit-length capacitance matrix $\mathbf{C}$. The transverse displacement current per unit length is simply the time derivative $\frac{\partial q_i(z, t)}{\partial t} = \sum_{j=1}^n C_{ij} \frac{\partial v_j(z, t)}{\partial t}$.

Dividing (2.22) by Δz and taking the limit $\Delta z \rightarrow 0$:

$$\frac{\partial i_i(z, t)}{\partial z} = - \sum_{j=1}^n G_{ij} v_j(z, t) - \sum_{j=1}^n C_{ij} \frac{\partial v_j(z, t)}{\partial t}. \quad (2.25)$$

Stacking all n equations for $i = 1, \dots, n$ into vector-matrix form directly produces the second Telegrapher equation:

$$\frac{\partial \mathbf{I}(z, t)}{\partial z} = -\mathbf{G}\mathbf{V}(z, t) - \mathbf{C} \frac{\partial \mathbf{V}(z, t)}{\partial t}. \quad (2.26)$$

2.4 Fundamental TEM Matrix Identities

A remarkable consequence of pure TEM wave propagation in a homogeneous dielectric medium (ϵ, μ) is that the electric and magnetic cross-sectional boundary-value problems are exact duals. Consequently, all TEM modes propagate at the speed of light in the

dielectric medium, $v = 1/\sqrt{\mu\epsilon}$, regardless of the multiconductor cross-sectional geometry. This leads to the fundamental matrix identity relating the inductance and capacitance matrices:

$$\mathbf{L}\mathbf{C} = \mathbf{C}\mathbf{L} = \mu\epsilon \mathbf{I}_n = \frac{1}{v^2} \mathbf{I}_n, \quad (2.27)$$

where $\mathbf{I}_n$ is the $n \times n$ identity matrix. Furthermore, if the host dielectric has uniform conductivity σ , the conductance matrix is related to the capacitance matrix by:

$$\mathbf{G} = \frac{\sigma}{\epsilon} \mathbf{C}. \quad (2.28)$$

Equations (2.27) and (2.28) illustrate how the macroscopic circuit matrices $\mathbf{L}$, $\mathbf{C}$, $\mathbf{G}$ are bound to the constitutive electromagnetic properties of the medium.

3 Coupled Multiconductor Telegrapher Equations in Matrix and Phasor Form

Combining (2.17) and (2.26), the spatial distribution and temporal dynamics of the voltage and current vectors along the multiconductor transmission line are governed by the coupled matrix system:

$$\frac{\partial \mathbf{V}(z, t)}{\partial z} = -\mathbf{R}\mathbf{I}(z, t) - \mathbf{L}\frac{\partial \mathbf{I}(z, t)}{\partial t}, \quad (3.1a)$$

$$\frac{\partial \mathbf{I}(z, t)}{\partial z} = -\mathbf{G}\mathbf{V}(z, t) - \mathbf{C}\frac{\partial \mathbf{V}(z, t)}{\partial t}. \quad (3.1b)$$

The system is specified by four $n \times n$ per-unit-length parameter matrices: inductance $\mathbf{L}$, resistance $\mathbf{R}$, conductance $\mathbf{G}$, and capacitance $\mathbf{C}$.

Assuming time-harmonic steady state with the $e^{j\omega t}$ phasor convention, where ω is the angular frequency and $\hat{\mathbf{V}}(z)$ and $\hat{\mathbf{I}}(z)$ are the phasor vectors, the time derivatives become $\frac{\partial}{\partial t} \rightarrow j\omega$, transforming the equations into:

$$\frac{d\hat{\mathbf{V}}(z)}{dz} = -\mathbf{Z}\hat{\mathbf{I}}(z) = -(\mathbf{R} + j\omega\mathbf{L})\hat{\mathbf{I}}(z), \quad (3.2a)$$

$$\frac{d\hat{\mathbf{I}}(z)}{dz} = -\mathbf{Y}\hat{\mathbf{V}}(z) = -(\mathbf{G} + j\omega\mathbf{C})\hat{\mathbf{V}}(z). \quad (3.2b)$$

Here, $\mathbf{Z} = \mathbf{R} + j\omega\mathbf{L}$ is the $n \times n$ per-unit-length series-impedance matrix (Ω/m), and $\mathbf{Y} = \mathbf{G} + j\omega\mathbf{C}$ is the $n \times n$ per-unit-length shunt-admittance matrix (S/m). The PUL units are Ω/m for $\mathbf{R}$, H/m for $\mathbf{L}$, S/m for $\mathbf{G}$, and F/m for $\mathbf{C}$.

4 Longitudinal Parameter Matrices (L and R)

Physical Meaning of the Per-Unit-Length Parameters

For a two-conductor line, the per-unit-length parameters have simple field-based definitions [1, p. 78]: the inductance $l = \psi/I$ is the magnetic flux per unit length linking the loop between the conductors divided by the loop current; the capacitance $c = q/V$ is the charge per unit length on the conductors divided by the voltage between them; and the conductance $g = J_t/V$ relates the transverse conduction current through the surrounding medium to the same voltage. These definitions assume perfect conductors and a homogeneous surrounding medium. The matrices $\mathbf{L}$, $\mathbf{C}$ and $\mathbf{G}$ of an $(n+1)$ -conductor line generalize them one-to-one, with ψ , q , J_t and the currents and voltages becoming n -vectors referenced to conductor 0.

Matrix Definitions

The longitudinal parameter matrices model the magnetic flux linkages and ohmic losses associated with current loops formed by the signal conductors and the reference conductor:

- **Per-Unit-Length Inductance Matrix, $\mathbf{L}$** ($n \times n$, symmetric positive definite for independent circuits): The inductance matrix contains the individual per-unit-length self-inductances, l_{ii} , of the circuits formed between conductor i and reference conductor 0 (i.e., ground), and the per-unit-length mutual inductances, $l_{ij} = l_{ji}$, between circuits i and j , as:

$$\mathbf{L} = \begin{bmatrix} l_{11} & l_{12} & \cdots & l_{1n} \\ l_{21} & l_{22} & \cdots & l_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix}. \quad (4.1)$$

- **Per-Unit-Length Resistance Matrix, $\mathbf{R}$** ($n \times n$, symmetric positive definite when all signal-conductor resistances are positive): The resistance matrix accounts for the finite conductivity of the conductors. The diagonal entries $(r_i + r_0)$ represent the total round-trip loop resistance for circuit i consisting of signal conductor resistance r_i and the common return conductor resistance r_0 . The off-diagonal entries r_0 represent the common-impedance resistive coupling due to shared return current in conductor 0:

$$\mathbf{R} = \begin{bmatrix} r_1 + r_0 & r_0 & \cdots & r_0 \\ r_0 & r_2 + r_0 & \cdots & r_0 \\ \vdots & \vdots & \ddots & \vdots \\ r_0 & r_0 & \cdots & r_n + r_0 \end{bmatrix}, \quad (4.2)$$

where r_i is the per-unit-length resistance of signal conductor i , and r_0 is the per-unit-length resistance of the reference conductor 0.

5 Transverse Parameter Matrices (G and C)

The transverse parameters represent the conduction and displacement currents flowing across the dielectric medium in the transverse cross-sectional plane:

- **Per-Unit-Length Conductance Matrix, $\mathbf{G}$** ($n \times n$, symmetric): Represents transverse conduction current (dielectric leakage losses). Let $g_{ij} = g_{ji} \geq 0$ be the conductance between signal conductors i and j , and let $g_{i0} \geq 0$ be the conductance between signal conductor i and the reference. Then:

$$\mathbf{G} = [G_{ij}], \quad G_{ij} = \begin{cases} g_{i0} + \sum_{\substack{k=1 \\ k \neq i}}^n g_{ik}, & i = j, \\ -g_{ij}, & i \neq j. \end{cases} \quad (5.1)$$

- **Per-Unit-Length Capacitance Matrix, $\mathbf{C}$** ($n \times n$, symmetric positive definite for a well-posed cross-section referenced to conductor 0): Represents transverse displacement current (electric-field coupling). Let $c_{ij} = c_{ji} \geq 0$ be the partial capacitance between signal conductors i and j , and let $c_{i0} \geq 0$ be the partial capacitance between signal conductor i and the reference. Then:

$$\mathbf{C} = [C_{ij}], \quad C_{ij} = \begin{cases} c_{i0} + \sum_{\substack{k=1 \\ k \neq i}}^n c_{ik}, & i = j, \\ -c_{ij}, & i \neq j. \end{cases} \quad (5.2)$$

Matrix structure For both transverse matrices, each diagonal entry is the sum of all branches incident on that conductor, including its branch to conductor 0; each off-diagonal entry is the negative of the mutual branch parameter. Thus the matrices have nonpositive off-diagonal entries and are diagonally dominant. The conductance matrix is positive semidefinite, and is positive definite when every connected component of the transverse conductance network has a path to the reference. The capacitance matrix is positive definite for a well-posed cross-section referenced to conductor 0.

6 Spatial Discretization, Line Segmentation, and Lumped Circuit Models

While the coupled Telegrapher equations (3.1) describe a continuous distributed-parameter system, general circuit simulators (such as SPICE or Modified Nodal Analysis engines) natively solve systems of ordinary differential equations (ODEs) representing lumped linear and nonlinear circuit elements. To simulate multiconductor transmission lines within transient circuit solvers, the continuous partial differential equations (PDEs) are spatially discretized into a cascade of discrete segments of finite length Δz .

6.1 Principles of Spatial Discretization

Consider a multiconductor line of physical length $\mathcal{L}$. We partition the line into N identical segments of length $\Delta z = \mathcal{L}/N$, with discrete spatial grid points located at $z_k = k \Delta z$ for $k = 0, 1, \dots, N$. Replacing the spatial derivatives $\partial/\partial z$ in the continuous Telegrapher equations with finite-difference approximations over the k -th interval $[z_k, z_{k+1}]$ yields:

$$\mathbf{V}(z_k, t) - \mathbf{V}(z_{k+1}, t) \approx \Delta z \mathbf{R} \mathbf{I}_k(t) + \Delta z \mathbf{L} \frac{d\mathbf{I}_k(t)}{dt}, \quad (6.1)$$

$$\mathbf{I}(z_k, t) - \mathbf{I}(z_{k+1}, t) \approx \Delta z \mathbf{G} \mathbf{V}_k(t) + \Delta z \mathbf{C} \frac{d\mathbf{V}_k(t)}{dt}. \quad (6.2)$$

In the frequency domain, each incremental section of length Δz is characterized by an $n \times n$ total series impedance matrix and an $n \times n$ total shunt admittance matrix:

$$\mathbf{Z}_\Delta = (\mathbf{R} + j\omega\mathbf{L}) \Delta z, \quad \mathbf{Y}_\Delta = (\mathbf{G} + j\omega\mathbf{C}) \Delta z. \quad (6.3)$$

Equations (6.1) and (6.2) can be synthesized into discrete lumped network topologies depending on how the series impedance $\mathbf{Z}_\Delta$ and shunt admittance $\mathbf{Y}_\Delta$ are allocated within each cell.

6.2 Canonical Lumped Equivalent Topologies: Π , T, and Half-T Models

Three standard lumped circuit configurations are commonly employed in transmission line simulation, as illustrated in Figure 3.

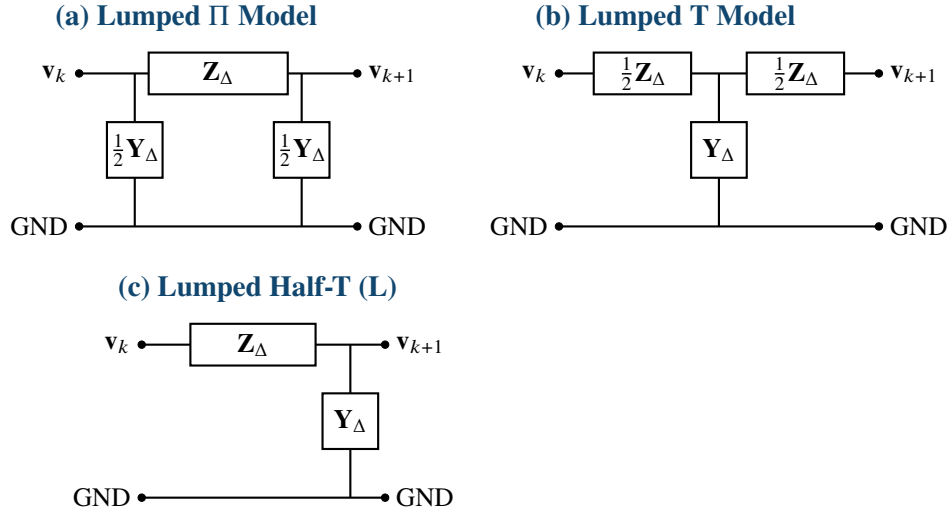


Figure 3. Canonical lumped-element cell configurations for an incremental line segment of length Δz : (a) Symmetrical Π cell, (b) Symmetrical T cell, and (c) Asymmetrical Half-T (inverted-L) cell.

- (a) **Lumped Π Model:** In the Π configuration, the full series impedance branch Z_{Δ} connects the input and output node vectors directly, while the shunt admittance is split symmetrically into two equal halves, $\frac{1}{2}Y_{\Delta}$, positioned at the input and output terminals:

$$\mathbf{Y}_{\text{in}} = \frac{1}{2}(\mathbf{G} + j\omega\mathbf{C}) \Delta z, \quad \mathbf{Z}_{\text{ser}} = (\mathbf{R} + j\omega\mathbf{L}) \Delta z, \quad \mathbf{Y}_{\text{out}} = \frac{1}{2}(\mathbf{G} + j\omega\mathbf{C}) \Delta z. \quad (6.4)$$

The transmission matrix (**ABCD** parameters) of a single Π cell is:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}_{\Pi} = \begin{bmatrix} \mathbf{I}_n + \frac{1}{2}\mathbf{Z}_{\Delta}\mathbf{Y}_{\Delta} & \mathbf{Z}_{\Delta} \\ \mathbf{Y}_{\Delta} \left(\mathbf{I}_n + \frac{1}{4}\mathbf{Z}_{\Delta}\mathbf{Y}_{\Delta} \right) & \mathbf{I}_n + \frac{1}{2}\mathbf{Y}_{\Delta}\mathbf{Z}_{\Delta} \end{bmatrix}. \quad (6.5)$$

Properties: The Π model is symmetric and exhibits second-order spatial accuracy $O(\Delta z^2)$. It is particularly favored when line terminations are capacitive or open-circuited, as the terminal shunt capacitors naturally merge into terminal nodal capacitance matrices without adding unnecessary voltage states.

- (b) **Lumped T Model:** In the T configuration, the shunt admittance $\mathbf{Y}_{\Delta}$ is placed in a single central branch connected to an internal midpoint node, while the series impedance is divided symmetrically into two equal halves on either side:

$$\mathbf{Z}_{\text{in}} = \frac{1}{2}(\mathbf{R} + j\omega\mathbf{L}) \Delta z, \quad \mathbf{Y}_{\text{sh}} = (\mathbf{G} + j\omega\mathbf{C}) \Delta z, \quad \mathbf{Z}_{\text{out}} = \frac{1}{2}(\mathbf{R} + j\omega\mathbf{L}) \Delta z. \quad (6.6)$$

The transmission matrix of a single T cell is:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}_T = \begin{bmatrix} \mathbf{I}_n + \frac{1}{2}\mathbf{Z}_\Delta \mathbf{Y}_\Delta & \mathbf{Z}_\Delta \left(\mathbf{I}_n + \frac{1}{4}\mathbf{Y}_\Delta \mathbf{Z}_\Delta \right) \\ \mathbf{Y}_\Delta & \mathbf{I}_n + \frac{1}{2}\mathbf{Y}_\Delta \mathbf{Z}_\Delta \end{bmatrix}. \quad (6.7)$$

Properties: The T model is the exact network dual of the Π model and also provides second-order spatial accuracy $O(\Delta z^2)$. It is advantageous when interconnects interface with low-impedance or inductive terminations.

- (c) **Lumped Half-T (Inverted-L) Model:** The Half-T cell employs an asymmetrical structure consisting of a single series impedance $\mathbf{Z}_\Delta$ followed by a single shunt admittance $\mathbf{Y}_\Delta$ (or vice versa):

$$\mathbf{Z}_{\text{ser}} = (\mathbf{R} + j\omega\mathbf{L}) \Delta z, \quad \mathbf{Y}_{\text{sh}} = (\mathbf{G} + j\omega\mathbf{C}) \Delta z. \quad (6.8)$$

The transmission matrix is:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}_{\text{Half-T}} = \begin{bmatrix} \mathbf{I}_n + \mathbf{Z}_\Delta \mathbf{Y}_\Delta & \mathbf{Z}_\Delta \\ \mathbf{Y}_\Delta & \mathbf{I}_n \end{bmatrix}. \quad (6.9)$$

Properties: A single Half-T cell has first-order spatial accuracy $O(\Delta z)$. However, when N identical Half-T cells are cascaded in series, adjacent internal branches combine, effectively synthesizing an internal T or Π cascade. Because each cell contains only one set of inductors and capacitors, it minimizes the total component count and state-variable matrix size in SPICE MNA netlists.

6.3 Alternative Cell Allocations and the Choice of Cell

The Π and T cells split the series and shunt elements symmetrically. The total elements of a cell of length Δz may, however, be distributed in other ways while still converging to the Telegrapher equations as $\Delta z \rightarrow 0$. Paul [1, p. 86] lists, for the two-conductor line, the symmetric Π cell (shunt $\frac{1}{2}c\Delta z$ and $\frac{1}{2}g\Delta z$ at each end), the symmetric T cell (series $\frac{1}{2}r\Delta z$ and $\frac{1}{2}l\Delta z$ at each end), and two asymmetric cells: one with shunt elements split $\frac{2}{3}-\frac{1}{3}$ between input and output, and one with series elements split $\frac{1}{4}-\frac{3}{4}$ around a single shunt branch. These are the Π -like and T-like cells of Figure 3 with different weights, and the symmetric choices are preferred in practice because the cascade of identical cells is then reciprocal and symmetric under reversal of the line direction, which the asymmetric choices break. For an n -conductor line the same allocations apply with r, l, g, c replaced by $\mathbf{R}, \mathbf{L}, \mathbf{G}, \mathbf{C}$.

Table 1. Correspondence between the matrices used here and the HSPICE U-model precomputed parameters [2, p. 416].

This document	HSPICE element	Meaning
l_{ii}, l_{ij} (L)	l11, l12, ...	self and mutual inductance per length
$c_{ij}, i \neq j$	c12, c13, ...	capacitance between conductors i and j
c_{i0}	cr1, cr2, ...	capacitance of conductor i to the reference
g_{ij}, g_{i0}	g12, ...; gr1, ...	dielectric-leakage conductances
r_{ii} (R)	$r_{is} + r_{ic}$	skin-effect and DC core resistance of line i
r_{00}	$r_{rs} + r_{rc}$	same for the reference conductor

6.4 The Multiconductor Lumped Cell as Used in Circuit Simulators

Circuit simulators implement the multiconductor cell directly from the PUL matrices. In the HSPICE U-model with precomputed parameters (ELEV=2), up to five signal conductors and a reference conductor are described by capacitance, conductance, inductance and resistance per unit length, which may come from a field solver, measurement, or vendor data [2, p. 416]. Each conductor carries two resistances, a high-frequency resistance accounting for skin effect and bent wires, and a DC core resistance, so the series resistance of conductor i is the sum $r_{is} + r_{ic}$ of a skin-effect term and a DC term. The complete cell is reproduced in Appendix A. The $n \times n$ matrices of this monograph map onto the simulator's parameters as listed in Table 1, and the structure of the three-conductor cell coincides with the symmetric T/Π cells above: each series branch of conductor i is split into two halves of $\frac{1}{2}l_{ii}$ with mutual inductive coupling $\frac{1}{2}l_{ij}$ between the corresponding halves, while the shunt branches c_{ij}, g_{ij} between conductors and c_{ri}, g_{ri} to the reference appear between the two halves.

Two practical consequences follow. First, a simulator cell is exactly the lumped cell of Figure 3 generalized to n conductors, so the sizing rules of Section 6 apply unchanged. Second, the reference conductor has its own series resistance, which a matrix **R** referenced to conductor 0 captures through the common term r_{00} added to every entry of **R** (see Section 8).

6.5 Rule of Thumb: Maximum Frequency Content of a Digital Pulse ($f_{\max}$)

In high-speed digital systems, transient signals are characterized by trapezoidal voltage pulses with finite 10%–90% rise and fall times t_r . To faithfully model signal propagation without truncating significant harmonic components, one must determine the maximum significant frequency $f_{\max}$ present in the pulse.

Fourier Spectral Roll-Off and Knee Frequency. The Fourier spectrum envelope $|X(f)|$ of a trapezoidal pulse with duration T and rise time t_r exhibits two characteristic break frequencies:

- (i) An initial -20 dB/decade ($1/f$) roll-off beginning at the pulse width break frequency $f_1 = 1/(\pi T)$.
- (ii) A steep -40 dB/decade ($1/f^2$) roll-off beyond the rise-time break frequency, commonly known as the *knee frequency* [4]:

$$f_{\text{knee}} = \frac{1}{\pi t_r} \approx \frac{0.318}{t_r} \quad \text{to} \quad \frac{0.5}{t_r}. \quad (6.10)$$

Beyond f_{knee} , the harmonic energy decays so rapidly that spectral components contribute negligibly to waveform distortion or peak voltage overshoot.

Standard Signal Integrity Engineering Rules of Thumb. Depending on the required modeling accuracy, two widely adopted rules of thumb govern f_{max} :

Thumb rules for the maximum frequency in a digital pulse

$$f_{\text{max}} \approx \frac{0.35}{t_r} \quad (\text{Standard 1st-order single-pole bandwidth equivalence}), \quad (6.11)$$

$$f_{\text{max}} \approx \frac{0.5}{t_r} \quad (\text{Conservative Signal Integrity knee-frequency rule}). \quad (6.12)$$

The $0.35/t_r$ relationship arises directly from the step response of a single-pole RC low-pass system, where the 10%–90% rise time is $t_r = \tau \ln(0.9/0.1) \approx 2.197\tau$, while the 3-dB bandwidth is $f_{-3\text{dB}} = 1/(2\pi\tau)$, giving $f_{-3\text{dB}} \cdot t_r \approx 2.197/(2\pi) \approx 0.35$.

6.6 Sizing the Number of Cascade Segments (N)

Once f_{max} is established from the fastest edge rate t_r , the required number of lumped segments N is determined by enforcing that each individual cell length Δz is electrically short relative to the minimum wavelength λ_{min} .

Wavelength Discretization Criterion. At frequency f_{max} , the minimum spatial wavelength along the multiconductor line is:

$$\lambda_{\text{min}} = \frac{v}{f_{\text{max}}} = \frac{1}{f_{\text{max}} \sqrt{\mu\epsilon}}, \quad (6.13)$$

where v is the phase velocity in the dielectric. For a lumped-element cascade to accurately approximate continuous wave propagation without introducing severe numerical dispersion

or false resonance cutoffs, the physical length of each segment must not exceed a small fraction of the minimum wavelength:

$$\Delta z \leq \frac{\lambda_{\min}}{10} \quad (\text{Standard engineering accuracy}), \quad (6.14)$$

$$\Delta z \leq \frac{\lambda_{\min}}{20} \quad (\text{High-fidelity / crosstalk analysis}). \quad (6.15)$$

Determining the Number of Segments N . Since the total line length is $\mathcal{L} = N \Delta z$, applying the criterion $\Delta z \leq \lambda_{\min}/10$ yields:

$$N = \frac{\mathcal{L}}{\Delta z} \geq \frac{10 \mathcal{L}}{\lambda_{\min}} = \frac{10 \mathcal{L} f_{\max}}{v}. \quad (6.16)$$

Defining the one-way time-of-flight (propagation delay) of the line as $\tau = \mathcal{L}/v = \mathcal{L}\sqrt{lc}$, we express N directly in terms of time-domain quantities:

$$N \geq 10 \tau f_{\max}. \quad (6.17)$$

Substituting the standard pulse bandwidth thumb rule $f_{\max} \approx 0.35/t_r$ from (6.11) into (6.17) provides the definitive time-domain sizing formula:

Thumb rule for the minimum number of segments N

$$N \geq \frac{3.5 \tau}{t_r} \approx 3.5 \times \frac{\text{Line One-Way Delay}}{\text{Pulse 10\%–90\% Rise Time}}. \quad (6.18)$$

Under the conservative knee-frequency criterion $f_{\max} \approx 0.5/t_r$ and $\Delta z \leq \lambda_{\min}/20$:

$$N \geq \frac{10 \tau}{t_r}. \quad (6.19)$$

Physical and Numerical Implications of Segment Sizing:

- **Electrically Short Lines** ($\tau \ll t_r/3.5$): When the propagation delay across the line is small compared to the rise time, the wave across the line is quasi-static. Setting $N = 1$ is sufficient, and the transmission line behaves essentially as a single lumped capacitive load or inductive choke.
- **Electrically Long Lines** ($\tau \geq t_r$): When the rise time is comparable to or smaller than the transit time, distributed propagation and reflection phenomena dominate. A lumped ladder has a cutoff frequency $f_c \approx 1/(\pi \Delta z \sqrt{lc}) = N/(\pi \tau)$. If too few segments are used ($N \lesssim \pi \tau f_{\max} \approx 1.1 \tau/t_r$), f_c falls below $f_{\max}$, causing severe numerical

ringing, unphysical Gibbs oscillations, and artificial rise-time degradation. The rule $N \geq 3.5 \tau/t_r$ keeps f_c roughly three times above $f_{\max}$.

- **Over-Segmentation Penalty:** Conversely, selecting N excessively large ($N \gg 20 \tau/t_r$) yields negligible gain in accuracy while drastically increasing the matrix dimension and slowing down transient SPICE time-stepping.

7 Numerical Example: Sizing and Segmenting a Single Line

This section applies the sizing rules of the previous section to a concrete case. For a single line ($n = 1$) the matrices $\mathbf{R}$, $\mathbf{L}$, $\mathbf{G}$, $\mathbf{C}$ reduce to the scalar per-unit-length parameters r , l , g and c , and $\mathcal{L}$ is the line length.

7.1 Problem Data

A 15 cm line is driven by the trapezoidal pulse sketched in Figure 4, through a source resistance R_s and into a load R_L . The data are collected in Table 2. The task is to find the maximum significant frequency of the pulse, decide the number of segments N , and compute the element values of the segmented model.

Table 2. Data of the numerical example.

Quantity	Symbol	Value
Line length	$\mathcal{L}$	15 cm
Resistance	r	8 Ω/m
Inductance	l	300 nH/m
Conductance	g	0.5 mS/m
Capacitance	c	120 pF/m
Pulse amplitude	V_0	1 V
Rise and fall time (10%–90%)	$t_r = t_f$	100 ps
Pulse width	T	2 ns
Source and load resistance	$R_s = R_L$	50 Ω

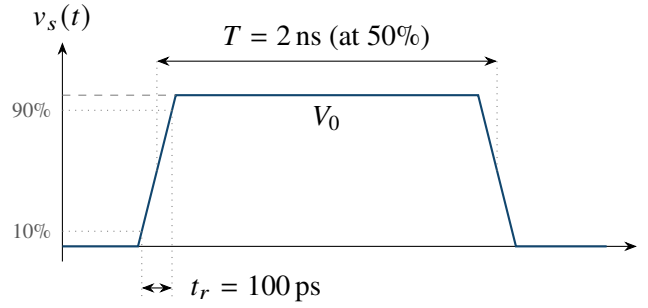


Figure 4. Trapezoidal test pulse (rise and fall times exaggerated, not to scale).

7.2 Step-by-Step Solution

Step 1: Line characteristics. The phase velocity, characteristic impedance and one-way delay follow from the per-unit-length parameters:

$$v = \frac{1}{\sqrt{lc}} = 1.667 \times 10^8 \text{ m/s} \approx 0.556 c, \quad (7.1)$$

$$Z_0 = \sqrt{\frac{l}{c}} = 50 \Omega, \quad \tau = \frac{\mathcal{L}}{v} = \frac{0.15 \text{ m}}{1.667 \times 10^8 \text{ m/s}} = 0.90 \text{ ns}. \quad (7.2)$$

The attenuation of a low-loss line is $\alpha \approx r/(2Z_0) + gZ_0/2 = 0.0925 \text{ Np/m}$, so the total loss is $\alpha\mathcal{L} = 0.0139 \text{ Np}$ (about 0.12 dB). At the highest frequency of interest $\omega l \approx 6.6 \times 10^3 \Omega/\text{m} \gg r$, so the line is nearly lossless and the sizing rules of Section 6 apply directly. Because $\tau/t_r = 9$, the line is electrically long and must be modeled as a distributed line.

Step 2: Maximum significant frequency. From the rise time, using (6.11) and (6.12):

$$f_{\max} \approx \frac{0.35}{t_r} = 3.5 \text{ GHz} \quad (\text{standard}), \quad f_{\max} \approx \frac{0.5}{t_r} = 5 \text{ GHz} \quad (\text{conservative}). \quad (7.3)$$

The knee frequency is $f_{\text{knee}} = 1/(\pi t_r) = 3.18 \text{ GHz}$, and the pulse-width break frequency is $f_1 = 1/(\pi T) = 159 \text{ MHz}$. The design value is $f_{\max} = 3.5 \text{ GHz}$.

Step 3: Number of segments. The minimum wavelength and the largest allowed segment length are

$$\lambda_{\min} = \frac{v}{f_{\max}} = \frac{1.667 \times 10^8 \text{ m/s}}{3.5 \text{ GHz}} = 47.6 \text{ mm}, \quad \Delta z \leq \frac{\lambda_{\min}}{10} = 4.76 \text{ mm}. \quad (7.4)$$

Hence, by (6.16) and (6.18),

$$N \geq \frac{\mathcal{L}}{\lambda_{\min}/10} = \frac{0.15 \text{ m}}{4.76 \text{ mm}} = 31.5, \quad N \geq \frac{3.5 \tau}{t_r} = 3.5 \times \frac{0.9 \text{ ns}}{0.1 \text{ ns}} = 31.5. \quad (7.5)$$

Rounding up gives $N = 32$ segments of length $\Delta z = \mathcal{L}/N = 4.6875 \text{ mm}$. For the conservative case, $f_{\max} = 5 \text{ GHz}$ gives $\lambda_{\min} = 33.3 \text{ mm}$ and $\Delta z \leq \lambda_{\min}/20 = 1.67 \text{ mm}$, that is $N \geq 10\tau/t_r = 90$ from (6.19). The standard value $N = 32$ is used below; $N = 90$ would be chosen for crosstalk or other high-fidelity studies.

Step 4: Element values of one segment. Each segment is a Π cell (6.4) with

$$\Delta R = r \Delta z, \quad \Delta L = l \Delta z, \quad \Delta G = g \Delta z, \quad \Delta C = c \Delta z, \quad (7.6)$$

and the shunt elements split into two halves at the cell ends. Neighbouring halves merge, so each of the $N - 1 = 31$ internal nodes carries the full ΔG and ΔC , and only the two end nodes carry half values. Table 3 gives the numbers, and Figure 5 shows the resulting model.

Table 3. Element values of the segmented model ($N = 32$, $\Delta z = 4.6875 \text{ mm}$).

Element	Per segment	End-node shunt	Total ($\times N$)
Series resistance ΔR	37.5 m Ω	—	1.2 Ω
Series inductance ΔL	1.406 nH	—	45 nH
Shunt conductance ΔG	2.344 μS	1.172 μS	75 μS
Shunt capacitance ΔC	0.5625 pF	0.2813 pF	18 pF

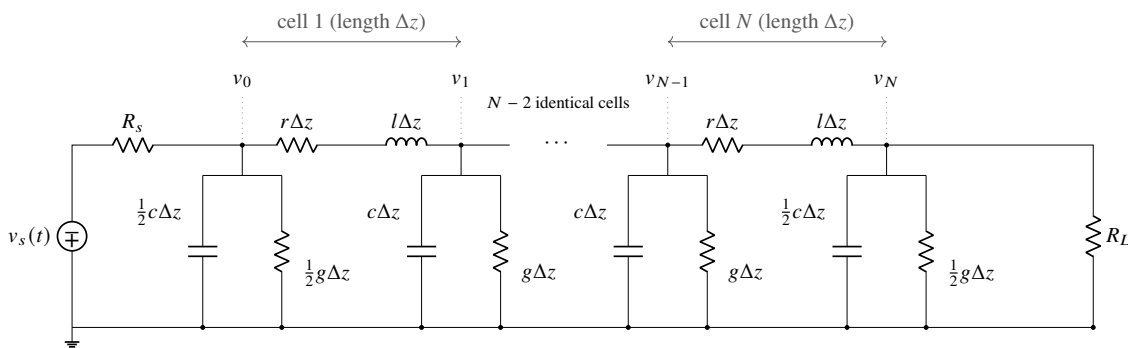


Figure 5. Segmented model of the single line: a cascade of $N = 32$ lumped Π cells between the source (v_s , R_s) and the load R_L . Series $r\Delta z$ and $l\Delta z$ form each cell; shunt $g\Delta z$ and $c\Delta z$ sit at the nodes, with half values at the two end nodes.

Step 5: Check of the choice. The ladder has a per-cell delay $\sqrt{\Delta L \Delta C} = 28.1 \text{ ps} = t_r/3.56$ and a cutoff frequency $f_c = 1/(\pi\sqrt{\Delta L \Delta C}) = N/(\pi\tau) = 11.3 \text{ GHz}$, which is $3.2 f_{\max}$. As a numerical check, the voltage transfer V_L/V_s was evaluated at $f_{\max}$ for the exact line and for N cascaded Π cells, with

$$\frac{V_L}{V_s} = \frac{1}{A + B/R_L + R_s (C + D/R_L)}, \quad (7.7)$$

where the exact line has $A = D = \cosh \gamma \mathcal{L}$, $B = Z_c \sinh \gamma \mathcal{L}$ and $C = \sinh(\gamma \mathcal{L})/Z_c$ with $\gamma = \sqrt{(r + j\omega l)(g + j\omega c)}$ and $Z_c = \sqrt{(r + j\omega l)/(g + j\omega c)}$, and the ladder uses the matrix of (6.4) raised to the power N . Table 4 summarizes the result.

Table 4. Lumped ladder versus exact line at $f_{\max} = 3.5 \text{ GHz}$ ($R_s = R_L = 50 \Omega$). The extra delay is the phase error divided by $360 f_{\max}$.

N	Δz (mm)	f_c (GHz)	Magnitude error (dB)	Extra delay (ps)
8	18.75	2.83	−88.2 (blocked)	—
16	9.375	5.66	−0.13	69.7
32	4.688	11.32	−0.016	15.0
90	1.667	31.83	−0.001	1.8

With $N = 8$ the cutoff $f_c = 2.83 \text{ GHz}$ lies below $f_{\max}$, so the highest harmonics of the pulse are blocked. With $N = 16$ the signal passes, but the ladder is slower than the line by 70 ps, which is 70% of the rise time. With the rule value $N = 32$ the extra delay is 15 ps (1.7% of τ , or $0.15 t_r$), and with $N = 90$ it falls to 1.8 ps. The magnitude is accurate in all cases above the cutoff, since the matched line has a flat response.

Result of the example $f_{\max} = 3.5 \text{ GHz}$, $\lambda_{\min} = 47.6 \text{ mm}$, $N = 32$ Π cells of $\Delta z = 4.6875 \text{ mm}$, each with $\Delta R = 37.5 \text{ m}\Omega$, $\Delta L = 1.406 \text{ nH}$, $\Delta G = 2.344 \text{ }\mu\text{S}$ and $\Delta C = 0.5625 \text{ pF}$ (half values at the two end nodes). The conservative criterion would require $N = 90$.

8 Numerical Example: A Three-Conductor Line from Per-Unit-Length Circuit Data

The second example goes the other way round: from a lumped per-unit-length circuit to the matrices $\mathbf{R}, \mathbf{L}, \mathbf{G}, \mathbf{C}$. The circuit in Figure 6 is the three-signal-conductor line with reference of Paul's exercise [1, Fig. P3.6, p. 109]. The values are textbook data chosen for the exercise and are not meant to describe a physical structure.

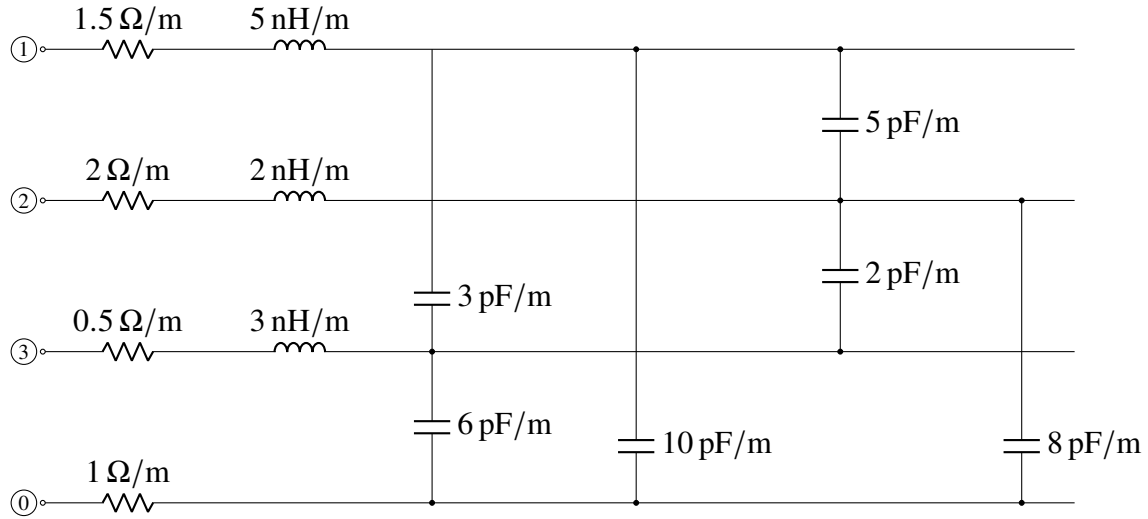


Figure 6. Per-unit-length circuit of a three-conductor line with reference conductor 0, after Paul [1, Fig. P3.6]. Capacitors are drawn between the conductors they connect; the mutual inductances are $l_{12} = 0.5 \text{ nH/m}$, $l_{13} = 1.5 \text{ nH/m}$ and $l_{23} = 1 \text{ nH/m}$. Crossing wires are not connected.

It is to be noted that the values given here are for the per-unit-length of the line. For the three-conductor line network as given in Figure 6 it holds that $c_{12} = 5 \text{ pF/m}$, $c_{13} = 3 \text{ pF/m}$, $c_{23} = 2 \text{ pF/m}$, $c_{r1} = 10 \text{ pF/m}$, $c_{r2} = 8 \text{ pF/m}$, $c_{r3} = 6 \text{ pF/m}$. The inductances of $\mathbf{L}$ are given directly and the resistances are $r_{1s} + r_{1c} = 1.5 \text{ } \Omega/\text{m}$, $r_{2s} + r_{2c} = 2 \text{ } \Omega/\text{m}$, $r_{3s} + r_{3c} = 0.5 \text{ } \Omega/\text{m}$ and $r_{rs} + r_{rc} = 1 \text{ } \Omega/\text{m}$. Because the model is a lumped cell, the segmentation rules of Section 7 determine how many cells are cascaded.

The circuit of Figure 6 is a lumped model of a line segment, and the values are per-unit-length. For a line of length $\mathcal{L}$, the matrices $\mathbf{R}$, $\mathbf{L}$, $\mathbf{G}$, $\mathbf{C}$ are multiplied by $\mathcal{L}$ to obtain the total series and shunt elements of the line.

N.B.: The example data for Figure 6 doesnot specify how each resistance divides between its skin-effect and DC parts, so the values given are the total resistances, not the individual skin-effect and DC components. When the skin-effect and DC resistances are specified separately, the total resistance is the sum of the two. It is to avoid confusion that the example uses the total resistances only, and the skin-effect and DC resistances are not specified seperately. Let it be clear that for a real line, the skin-effect and DC resistances are computed from the conductor geometry and material properties, and the total resistance is their sum. For example, if the skin-effect resistance is $1 \text{ } \Omega/\text{m}$ and the DC resistance is $0.5 \text{ } \Omega/\text{m}$, the total resistance is $1.5 \text{ } \Omega/\text{m}$. As an industrial case as shown in the HSPICE U-model of Table 1, the skin-effect and DC resistances are set to realize the total resistance $\mathbf{R}$.

8.1 Extraction of the Matrices

Inductance. The self inductances are $l_{11} = 5 \text{ nH/m}$, $l_{22} = 2 \text{ nH/m}$, $l_{33} = 3 \text{ nH/m}$ and the mutual inductances $l_{12} = 0.5 \text{ nH/m}$, $l_{13} = 1.5 \text{ nH/m}$, $l_{23} = 1 \text{ nH/m}$, so

$$\mathbf{L} = \begin{bmatrix} 5 & 0.5 & 1.5 \\ 0.5 & 2 & 1 \\ 1.5 & 1 & 3 \end{bmatrix} \text{ nH/m.} \quad (8.1)$$

Its eigenvalues are 1.33 nH/m , 2.65 nH/m and 6.02 nH/m , all positive, so $\mathbf{L}$ is positive definite as required for a passive line.

Capacitance. Reading the circuit gives the inter-conductor capacitances $c_{12} = 5 \text{ pF/m}$, $c_{13} = 3 \text{ pF/m}$, $c_{23} = 2 \text{ pF/m}$ and the capacitances to the reference $c_{10} = 10 \text{ pF/m}$, $c_{20} = 8 \text{ pF/m}$, $c_{30} = 6 \text{ pF/m}$. The diagonal entries of $\mathbf{C}$ are the sums of all capacitances attached to the conductor and the off-diagonal entries are the negatives of the mutual capacitances:

$$\mathbf{C} = \begin{bmatrix} c_{10} + c_{12} + c_{13} & -c_{12} & -c_{13} \\ -c_{12} & c_{20} + c_{12} + c_{23} & -c_{23} \\ -c_{13} & -c_{23} & c_{30} + c_{13} + c_{23} \end{bmatrix} = \begin{bmatrix} 18 & -5 & -3 \\ -5 & 15 & -2 \\ -3 & -2 & 11 \end{bmatrix} \text{ pF/m.} \quad (8.2)$$

The eigenvalues 7.7 pF/m , 14.4 pF/m and 21.9 pF/m are positive, and the matrix is hyperdominant (negative off-diagonal terms with positive row sums c_{10}, c_{20}, c_{30}), a sufficient condition for positive definiteness [1, p. 109].

Resistance. Because all return currents flow through conductor 0, the resistance matrix referenced to it contains the common term $r_{00} = 1 \Omega/\text{m}$ in every entry:

$$\mathbf{R} = \begin{bmatrix} r_1 + r_0 & r_0 & r_0 \\ r_0 & r_2 + r_0 & r_0 \\ r_0 & r_0 & r_3 + r_0 \end{bmatrix} = \begin{bmatrix} 2.5 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1.5 \end{bmatrix} \Omega/\text{m.} \quad (8.3)$$

Conductance. No conductances appear in the circuit, so $\mathbf{G} = \mathbf{0}$. If the line were embedded in a homogeneous medium of permittivity ε and conductivity σ , the TEM identity $\mathbf{G} = (\sigma/\varepsilon)\mathbf{C}$ of the matrix-identities subsection would apply.

8.2 Segmented Model

For a line of length $\mathcal{L}$ and N segments, every cell of length $\Delta z = \mathcal{L}/N$ uses the matrices above multiplied by Δz ; for instance, $\mathbf{C}\Delta z$ and $\mathbf{L}\Delta z$ are the shunt and series blocks of

the Π cell of Figure 3, and in the HSPICE U-model of Table 1 the same numbers enter as $c_{12} = 5 \text{ pF/m}$, $c_{r1} = c_{10}$, and so on, and the skin-effect and DC resistances are set to realize **R**. The segment count follows from the pulse bandwidth exactly as in Section 7.

9 Worked Example: Geometry of a Coupled Stripline Pair

The previous examples started from circuit values. This section starts one step earlier, from the physical structure, and follows it to the per-unit-length matrices. The structure is a pair of identical, parallel signal traces embedded in a homogeneous dielectric between two ground planes (an edge-coupled symmetric *stripline*). Both planes together form the reference conductor 0, so the line is a three-conductor system with $n = 2$ signal conductors, and the TEM identities of Section 6 hold exactly because the dielectric is homogeneous. Stripline geometries and their design equations are discussed in [4–6].

9.1 Line Geometry and Material Data

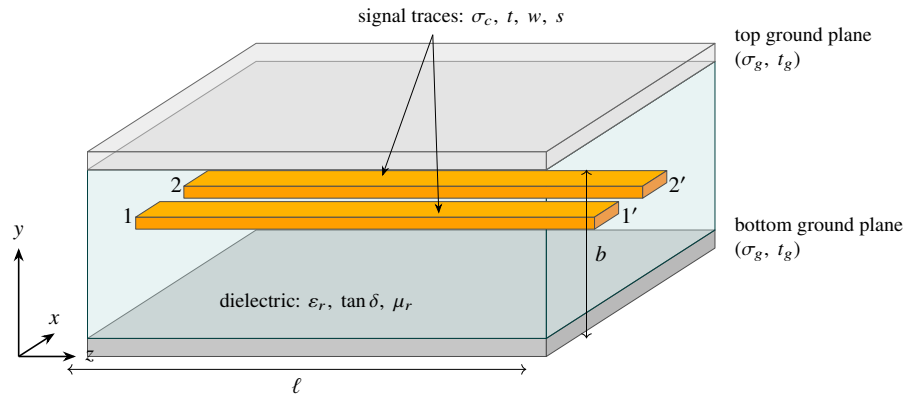
Figure 7 shows the structure: a three-dimensional view with the propagation axis z , a cross section drawn to scale, and the plan view of the two traces. All dimensions and material properties are collected in Table 5; they are representative of a standard FR-4-class printed circuit board with one-ounce copper.

9.2 From Geometry to the Per-Unit-Length Matrices

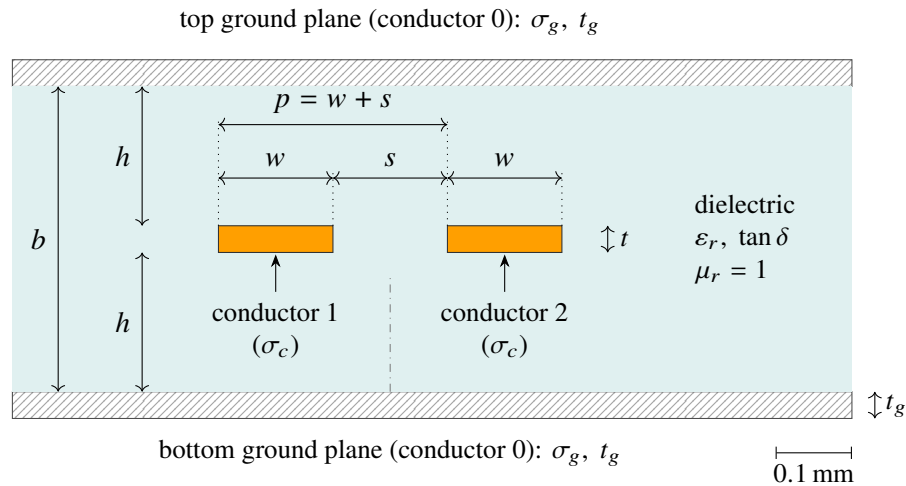
Capacitance and inductance. The cross section was solved as a two-dimensional electrostatic problem: Laplace's equation was solved by finite differences on a $2.5 \mu\text{m}$ grid with the two planes and the side walls (10 mm apart) at ground potential and the traces as equipotential rectangles of finite thickness. The capacitance matrix follows from the field energy for unit potentials on each trace. For a homogeneous dielectric the inductance matrix is obtained from the air-filled capacitance matrix $\mathbf{C}_0$, $\mathbf{L} = \mu_0 \epsilon_0 \mathbf{C}_0^{-1}$, so that $\mathbf{LC} = \mu_0 \mu_r \epsilon_0 \epsilon_r \mathbf{I}_n$. The result is

$$\mathbf{C} = \begin{bmatrix} 149.3 & -21.7 \\ -21.7 & 149.3 \end{bmatrix} \text{pF/m}, \quad \mathbf{L} = \begin{bmatrix} 319.8 & 46.4 \\ 46.4 & 319.8 \end{bmatrix} \text{nH/m}. \quad (9.1)$$

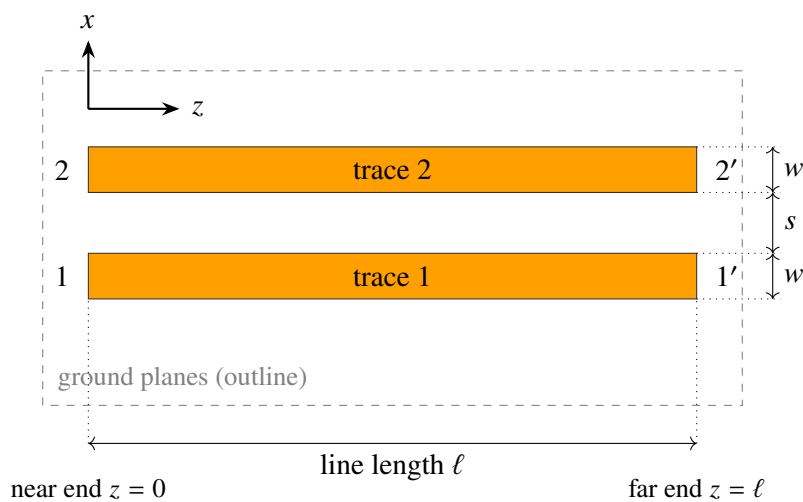
The coupling coefficients $-C_{12}/C_{11}$ and L_{12}/L_{11} are both 0.145, as the TEM identity requires. In terms of lumped elements, the capacitance between the traces is $c_{12} = 21.7 \text{ pF/m}$ and the capacitance of each trace to the reference is $c_{r1} = c_{r2} = C_{11} + C_{12} = 127.6 \text{ pF/m}$ (Appendix A).



(a) Three-dimensional view (vertical scale exaggerated).



(b) Cross section in the xy plane, drawn to scale (1 cm in the drawing equals 0.1 mm).



(c) Plan view (xz plane) of the two traces.

Figure 7. Geometry of the symmetric edge-coupled stripline pair: (a) perspective, (b) cross section to scale, (c) plan view. Both ground planes are one reference conductor, shorted together at a common potential.

Table 5. Geometry and material data of the coupled stripline pair.

Quantity	Symbol	Value
<i>Geometry</i>		
Line length	ℓ	150 mm
Trace width	w	0.150 mm (150 μm)
Edge-to-edge spacing	s	0.150 mm
Pitch (centre-to-centre)	$p = w + s$	0.300 mm
Trace thickness (1 oz Cu)	t	35 μm
Plane-to-plane dielectric height	b	0.400 mm
Trace-to-plane distance (centred)	$h = (b - t)/2$	0.1825 mm
Ground-plane thickness	t_g	35 μm
<i>Dielectric (FR-4 class), at 1 GHz</i>		
Relative permittivity	ϵ_r	4.2
Loss tangent	$\tan \delta$	0.02
Relative permeability	μ_r	1 (non-magnetic)
Equivalent conductivity	$\sigma_d = \omega \epsilon_0 \epsilon_r \tan \delta$	$4.7 \times 10^{-3} \text{ S/m}$
<i>Conductors (copper)</i>		
Conductivity of traces	σ_c	$5.8 \times 10^7 \text{ S/m}$
Conductivity of ground planes	σ_g	$5.8 \times 10^7 \text{ S/m}$
Skin depth at 1 GHz	$\delta_s = 1/\sqrt{\pi f \mu_0 \sigma_c}$	2.09 μm

Resistance. The DC resistance of a trace is $r_{\text{dc}} = 1/(\sigma_c w t) = 3.28 \Omega/\text{m}$. The two planes together carry the return current over a width of several plane spacings, which gives a reference resistance of the order of $0.1 \Omega/\text{m}$, small enough to neglect here, so $\mathbf{R} \approx r_{\text{dc}} \mathbf{I}_2$. At 1 GHz the skin depth is far smaller than the trace thickness, the current flows on the surface, and the resistance rises roughly to $1/(2\sigma_c \delta_s w) \approx 27 \Omega/\text{m}$ when current crowding at the edges is ignored. This frequency-dependent increase is what the high-frequency resistance r_{is} of the U-model represents.

Conductance. The loss tangent turns the dielectric into a conductor with $\sigma_d = \omega \epsilon_0 \epsilon_r \tan \delta$, so the identity $\mathbf{G} = (\sigma_d/\epsilon) \mathbf{C}$ becomes $\mathbf{G} = \omega \tan \delta \mathbf{C}$. At 1 GHz,

$$\mathbf{G} = \begin{bmatrix} 18.8 & -2.72 \\ -2.72 & 18.8 \end{bmatrix} \text{ mS/m.} \quad (9.2)$$

9.3 Modal Properties and Segmentation

Because the structure is symmetric, the line decouples into an even and an odd mode with

$$Z_e = \sqrt{\frac{L_{11} + L_{12}}{C_{11} + C_{12}}} = 53.6 \, \Omega, \quad Z_o = \sqrt{\frac{L_{11} - L_{12}}{C_{11} - C_{12}}} = 40.0 \, \Omega, \quad (9.3)$$

that is, a differential impedance $2Z_o = 80 \, \Omega$. Both modes travel at $v = 1/\sqrt{\mu_0\mu_r\epsilon_0\epsilon_r} = 1.463 \times 10^8 \, \text{m/s}$, a delay of 6.84 ps/mm and 1.03 ns over the full length. As a check, the single-trace stripline formula of IPC-2141A [3], $Z_0 = \frac{60}{\sqrt{\epsilon_r}} \ln \frac{4b}{0.67\pi(0.8w+t)} = 46.6 \, \Omega$, agrees with the mean $\frac{1}{2}(Z_e + Z_o) = 46.8 \, \Omega$ of the numerical modes to within 0.5 %.

For a 100 ps edge the rules of Section 6 give $f_{\max} \approx 0.35/t_r = 3.5 \, \text{GHz}$, $\lambda_{\min} = v/f_{\max} = 41.8 \, \text{mm}$, and

$$N \geq 10 \frac{\ell}{\lambda_{\min}} = 10 \cdot \frac{150}{41.8} = 35.9 \Rightarrow N = 36 \quad (9.4)$$

cells of length $\Delta z = 4.17 \, \text{mm}$, each built from the matrices above multiplied by Δz . The cell is the one-conductor-pair version of Figure 3 and, for the HSPICE U-model, uses the element values $c_{12} = 21.7 \, \text{pF/m}$, $c_{r1} = c_{r2} = 127.6 \, \text{pF/m}$ and the entries of $\mathbf{L}$ above, as described in Appendix A.

A The HSPICE U-Model with Precomputed Parameters

This appendix reproduces, in the notation of this monograph, the lumped multiconductor cell used by the HSPICE *U element* when the model is specified through precomputed per-unit-length parameters (ELEV=2). The description follows the simulator documentation [2, App. B, p. 416]; the schematic has been redrawn rather than copied.

A.1 Parameter Set

The precomputed form describes up to five signal conductors together with one reference conductor. The parameters may come from a field solver, from laboratory measurements, or from packaging specifications supplied by a vendor. Four groups are supplied:

- **Capacitance per length:** every conductor has a capacitance to each of the other conductors.
- **Conductance per length:** every conductor has a conductance to each of the other conductors, representing dielectric leakage.
- **Inductance per length:** every conductor has a self inductance and a mutual inductance to each of the other conductors.
- **Resistance per length:** every conductor has two resistances, a high-frequency resistance that accounts for skin effect and bent wires, and a DC core resistance.

A.2 Circuit Structure

Figure 8 shows the cell for three signal conductors and a reference conductor. It has the symmetric T-type structure of Section 6: on each side, the series branch of conductor i consists of half of the resistance, $\frac{1}{2}(r_{is} + r_{ic})$, in series with half of the self inductance, $\frac{1}{2}l_{ii}$, and the inductors of different conductors are coupled by the mutual terms $\frac{1}{2}l_{ij}$. In the middle, the shunt branches form the full set of inter-conductor capacitances c_{ij} and conductances g_{ij} , together with c_{ri} and g_{ri} from each conductor to the reference conductor. The reference conductor carries its own series resistance $\frac{1}{2}(r_{rs} + r_{rc})$ at each end and is tied to the simulator ground at the centre of the cell.

A.3 Element Count and Naming

The element names follow a fixed pattern. For a cell with n signal conductors the simulator needs:

- n pairs (r_{is}, r_{ic}) for the signal conductors and one pair (r_{rs}, r_{rc}) for the reference;

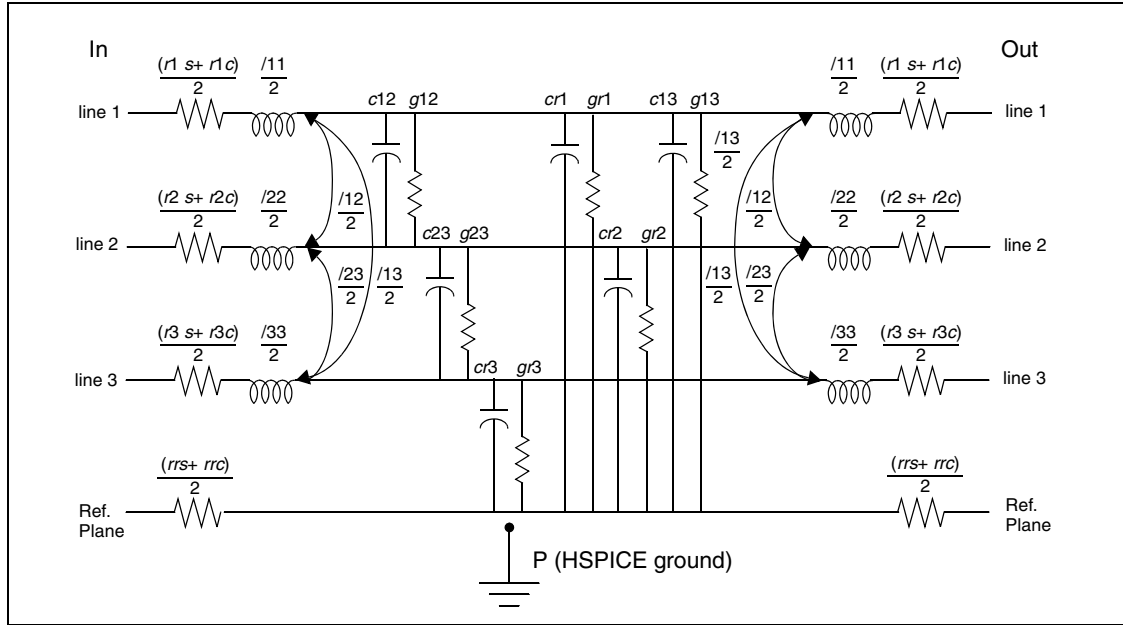


Figure 8. Precomputed U-model cell (ELEV=2) for three signal conductors and a reference conductor, redrawn after Synopsys [2, Fig. 113, p. 416]. Each conductor carries half of its series resistance and inductance at either end of the cell, with mutual coupling $\frac{1}{2}l_{ij}$ between the inductor halves, and the full shunt network in the middle. Crossing wires are not connected.

- $n(n-1)/2$ conductor-to-conductor capacitances c_{ij} and the same number of conductances g_{ij} ($i < j$);
- n capacitances c_{ri} and n conductances g_{ri} to the reference (the names cri and gri);
- $n(n+1)/2$ inductances l_{ij} ($i \leq j$), each appearing as a half value at both ends of the cell.

For $n = 3$ this is $6 + 3 + 3 + 3 + 3 + 6 = 24$ distinct values besides the reference resistances. The documentation lists the names for up to five conductors in its Figure 114 (p. 417) [2], which extends this pattern by index and is not reproduced here.

A.4 Relation to the PUL Matrices

With the matrices of Sections 6 and 8, the precomputed parameters correspond to the entries as follows:

$$\mathbf{L}_{ij} = l_{ij}, \quad (\text{A.1})$$

$$\mathbf{C}_{ii} = c_{ri} + \sum_{j \neq i} c_{ij}, \quad \mathbf{C}_{ij} = -c_{ij} \quad (i \neq j), \quad (\text{A.2})$$

$$\mathbf{G}_{ii} = g_{ri} + \sum_{j \neq i} g_{ij}, \quad \mathbf{G}_{ij} = -g_{ij} \quad (i \neq j), \quad (\text{A.3})$$

$$\mathbf{R}_{ii} = (r_{is} + r_{ic}) + (r_{rs} + r_{rc}), \quad \mathbf{R}_{ij} = r_{rs} + r_{rc} \quad (i \neq j). \quad (\text{A.4})$$

The last line expresses that the return current of every signal conductor flows through the common reference resistance. Conversely, to enter a computed set of matrices into the simulator, the mutual capacitances are $c_{ij} = -\mathbf{C}_{ij}$, the capacitances to the reference are the row sums $c_{ri} = \sum_j \mathbf{C}_{ij}$, and the same applies to $\mathbf{G}$.

References

- [1] C. R. Paul, *Analysis of Multiconductor Transmission Lines*, 2nd ed. Hoboken, NJ, USA: Wiley-IEEE Press, 2008.
- [2] Synopsys, Inc., *HSPICE User Guide: Signal Integrity Modeling and Analysis*, Version H-2013.03. Mountain View, CA, USA, 2013, App. B, “Lumped Transmission Line — U-Model,” p. 416.
- [3] IPC, *IPC-2141A: Design Guide for High-Speed Controlled Impedance Circuit Boards*. Bannockburn, IL, USA: IPC, 2004.
- [4] D. M. Pozar, *Microwave Engineering*, 4th ed. Hoboken, NJ, USA: John Wiley & Sons, Inc., 2011, ch. 2, pp. 49–57.
- [5] S. H. Hall and H. L. Heck, *Advanced Signal Integrity for High-Speed Digital Designs*. Hoboken, NJ, USA: John Wiley & Sons, Inc., 2009, ch. 3–4.
- [6] E. Bogatin, *Signal and Power Integrity – Simplified*, 3rd ed. Boston, MA, USA: Prentice Hall, 2018, ch. 7–8.